

Real-time tunneling

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Abstract

Starting with the equivalence of the rest energy of a particle to an amount of the radiant energy characterized by a frequency, in addition to the usual relativistic transformation rules leading to the wave-particle duality, we investigate the case in which this frequency is an internal property of the particle. This kind of interpretation of the frequency is shown to be relevant to the tunneling effect. The investigations in this direction yield (1) a purely real time everywhere, (2) an anti-hermitian momentum operator, (3) a corpuscular structure for the particle, and (4) all of the known theoretical predictions about the tunneling effect.

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1 Introduction

Quantum mechanical tunneling has become one of the most interesting applications of the quantum theory since its early days. It has proven to be the key concept in understanding the physics of the systems having classical stable configurations separated by an impenetrable energy barrier. Tunneling phenomenon is unique to quantum theory and play an important role in understanding the various physical processes, ranging from birth of the universe from the nothing [1] to α -decay [2].

In this work we restrict ourselves to simple quantum systems which can be analyzed by the use of the Schroedinger theory. In particular, we are not interested in the tunneling phenomena in field theories where one faces with an infinite number of degrees of freedom (See the recent review [3]). In this sense, analysis of the tunneling process is a text book example (See, for example, [4] or [5]). In the discussion of the stationary tunneling processes where a particle of energy E is incident on a potential barrier $V(x)$, entire spatial range is classified according to whether $E \geq V(x)$ or $E \leq V(x)$. While the former defines the normal region, the latter defines the barrier region where the total energy of the particle is below its potential energy. In the normal region particle has real-time trajectories satisfying the usual Newtonian laws of motion, whereas in the barrier region strict energy conservation forces particle to have imaginary-time trajectories [6, 7, 8]. As is well-known, in the barrier region Schroedinger equation has evanescent-wave solutions characterized by decaying and growing exponentials. Despite the well-understood formalism of the tunneling summarized here, the debate about the tunneling time continues to exist. This mainly stems from the fact that time is a parameter in the quantum theory, and consequently, one does not have an agreed-upon formalism to calculate it [9].

It would be convenient to use every opportunity to have a better insight into the tunnelling phenomenon as it might be useful in arriving at the full solution (including the tunneling time itself) of the problem. In this work our aim is not to calculate the tunneling time. We shall just present a different way of analyzing the tunneling effect to gain more insight into its nature. The method of the analysis can be summarized by mentioning that we introduce different Hamiltonians for normal and barrier regions; in particular particle is endowed with a negative kinetic energy in the barrier region. As will be seen in the text negative kinetic energy is not an ad hoc assumption, instead,

it will be shown to follow from the first principles. The main results of the work can be summarized as the reality of the time everywhere, non-hermitian character of the momentum in the barrier region, corpuscular nature of the particle in the barrier, and the reproduction of all of the known results.

In Sec.2 we mention the formalism and discuss its phenomenological viability in the classical and quantum limits.

In Sec.3 we present a detailed discussion of the results and compare them with those of the usual formalism.

2 Derivations

Let us consider a relativistic particle of rest mass m_0 . As usual, one can equate its rest energy m_0c^2 to a unit of Planckian energy [10, 11]

$$m_0c^2 = \hbar\omega_0 \tag{1}$$

where the emphasis is on the quantum character of this equation as it loses its meaning as $\hbar \rightarrow 0$. In what follows we require (1) to be covariant under Lorentz transformations, that is, it will hold in an arbitrary inertial frame K moving with velocity v relative to the rest frame K_0 in which (1) holds.

To determine the form of (1) in the moving frame K one should know the transformation rule for the frequency and mass. Actually, knowing the transformation rule for either of them is sufficient as long as the equality (1) holds in any inertial frame. Here we shall base discussion on the transformation properties of the frequency, and conclude that of the mass from the covariance. There are two known options one can follow in specifying the transformation rule for the frequency [12]. First, in K_0 there can be a plane wave $e^{i\omega_0 t_0}$ accompanying or co-existing with the particle. As the phase of a plane wave is a relativistic invariant, in the moving frame K equation (1) becomes a four-vector relation:

$$P_\mu = \hbar k_\mu \tag{2}$$

which is nothing but the usual relation between energy-momentum four-vector $P_\mu = (mc^2, c\vec{P})$ and the wave-vector $k_\mu = (\omega, \vec{k})$. It is in this sense that the particle is associated with a wave motion having the propagation

vector k_μ . The expression for m is the usual one

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \quad (3)$$

where m_0 is related to ω_0 via (1). The de Broglie relation (2) is the basic statement of the wave-particle duality allowing for the representation of the quantum mechanical particles by the wave phenomena, leading to wave mechanics.

Until now we have rederived the known statement of the wave-particle duality (2) from (1) by making the standard assumption that the particle is accompanied by a wave propagation. However, this transformation property of the frequency is just one alternative; that is, ω_0 in (1) can well be an internal property of the particle itself, in which case it is viewed as a clock of frequency ω_0 in K_0 . After ascribing this meaning to ω_0 , in the moving frame K equation (1) takes the form

$$M = \hbar\Omega . \quad (4)$$

Being one of the most fundamental results of the theory of relativity moving clocks slow down, that is $\Omega = \omega_0\sqrt{1 - v^2/c^2}$, so the moving frame mass M is related to its rest frame value via

$$M = m_0\sqrt{1 - v^2/c^2}. \quad (5)$$

Now let us remark on some consequences of this transformation rule. Equation (4), as opposed to (2), is not a four-vector relation, instead it just reexpresses (1) in the moving frame K , and there is no propagation characteristics (like \vec{k} in (2)) whatsoever in it. Hence, it must be emphasized that with this interpretation of ω_0 in (1) particle could be viewed as a corpuscle having no wave-like properties. Here, by corpuscle, we mean a localized object where the degree of localization varies from a point (classical material point) to a distribution (quantum distribution).

Furthermore, the transformation rule for mass in (5), unlike the one in (3), is completely unrealistic as it contradicts with the well-established experimental facts. In fact, it was merely because of this reason that de Broglie eliminated this transformation rule. Fortunately, however, the strange transformation rule (5) will be seen to match the requirements of the quantum tunneling.

Taking the non-relativistic limit of equations (3) and (5), using the usual formula for kinetic energy $(m - m_0)c^2$, and adding an arbitrary potential function $V(x)$ to take into account the possible interaction of the particle with the environment, one gets the following energy functions

$$(3) \rightsquigarrow h = \frac{1}{2}m_0\left(\frac{dx}{dt}\right)^2 + V(x) \quad (6)$$

$$(5) \rightsquigarrow H = -\frac{1}{2}m_0\left(\frac{dx}{dt}\right)^2 + V(x) \quad (7)$$

We note that while kinetic energy of the particle is positive-definite in the wave representation (6), it is negative-definite in the case of corpuscular representation (7). Furthermore, as the basic equality (1) guarantees, the energy functions (6) and (7) become identical if the velocity vanishes in both representation. We now analyze (6) and (7) in detail to investigate their physical implications.

2.1 Classical Considerations

Let us start discussing (6). For a time-independent potential, $V(x)$, energy is conserved, and $h = E$ is a first integral of the motion. Therefore the trajectory of the particle can be shown to satisfy

$$m_0\frac{dx}{dt} = \sqrt{2m_0(E - V(x))} \quad (8)$$

$$m_0\frac{d^2x}{dt^2} = -\frac{dV}{dx} . \quad (9)$$

The first equation requires $E \geq V(x)$ for the particle to have a real momentum. The next one is the well-known statement of the Newtonian laws of motion. As a result, the energy function (6) leads us to the well-known description of the classical systems for which the total energy of the system exceeds or at most equals the potential energy field on it, as is always the case.

Repeating the same steps of analysis for (7), with $H = \mathcal{E}$ is a constant of the motion, one obtains

$$m_0\frac{dx}{dt} = \sqrt{2m_0(V(x) - \mathcal{E})} \quad (10)$$

$$m_0\frac{d^2x}{dt^2} = \frac{dV}{dx} \quad (11)$$

which have entirely different implications on the particle dynamics than (8) and (9). The first equation requires $\mathcal{E} \leq V(x)$, for particle's momentum be real. The next equation, on the other hand, forces practically particle to obey Newtonian laws for an inverted potential $V(x) \rightarrow -V(x)$, that is, one does not have the true laws of motion for the trajectory of the particle. Consequently, the energy function H in (7) restricts particle to those regions of the space in which the total energy of the particle is at most equal to the potential energy on it.

If we require, for both h (6) and H (7), particle be endowed with real position and time coordinates we should restrict these energy functions to their appropriate regions of validity, that is, we should use h when $E \geq V(x)$, and H when $E \leq V(x)$ for a particle with total energy E . With this prescription in mind, one would recall the tunneling effect where such distinctions as $E \geq V(x)$ and $E \leq V(x)$ are of fundamental importance for the analysis of the problem. In fact, all these unusual properties (10) and (11) have already been observed and utilized in the analysis of the quantum mechanical tunneling by path integral methods [6, 8]. One should note, however, there is a fundamental distinction between the two; namely, (10) and (11) have been derived without introducing an Euclidean time, unlike the path integral formulation of the quantum tunneling where rotation of the time parameter from real to purely imaginary values is indispensable.

To combine the implications of wave- and corpuscular- representations we have analyzed till now, it would be convenient to refer a typical potential energy graph, like the one shown in Fig.1. In drawing Fig.1 we assumed a particle with coordinate x and total energy E . The points labelled by a , b and c are the classical turning points in which $V(x) = E$. In the discussion below we assume both energies E in (8) and \mathcal{E} in (10) are identical and equal to E , as suggested by Fig. 1.

For $a \leq x \leq b$ we have $E \geq V(x)$, so we conclude that in this region the total energy of the system has the expression in (6), and dynamical evolution of the particle's coordinate is governed by (9). The boxed label h stands for the energy function h in (6). While the velocity of the particle (8) vanishes as it approaches to the turning points a or b , it is accelerated through the equation of motion (9) back to the region of incidence. In this way, under the strict energy conservation, particle moves back and forth between the turning points a and b . The half-period, T , of the motion can be found [13]

by inverting (8) for time

$$T_h = m_0 \int_a^b \frac{dx}{\sqrt{2m_0(E - V(x))}}. \quad (12)$$

For $b \leq x \leq c$, however, we have $E \leq V(x)$ so we conclude that here the total energy of the system has to have the expression in (7), and dynamical evolution of the particle's coordinate is to be governed by (11). The boxed label H stands for the energy function H in (7). While the velocity of the particle (10) vanishes as it approaches to the turning points b or c , it is accelerated through the equation of motion (11) back to the region of incidence. In this way, under the strict energy conservation, particle moves back and forth between the turning points b and c . The half-period, T_H , of the motion can be found by inverting (10) for time

$$T_H = m_0 \int_b^c \frac{dx}{\sqrt{2m_0(V(x) - E)}}. \quad (13)$$

We conclude from these discussions that particle oscillates indefinitely whichever region it happens to fall initially. It remains trapped in the associated region unless the configuration of potential energy is modified by some external agent. From the realistic point of view, the h -region $a \leq x \leq b$ is the one where classical systems can exist, as is evidenced by the appearance of the usual equation of motion (9). The trajectory of the particle in H -region, $b \leq x \leq c$ is no way classical as dictated by its equation of motion (11). Although form of (11) is exactly the one one would use in analyzing the tunneling effect by path integral methods, its derivation does not rest on a Wick rotation of the time parameter; it follows directly from the energy function H (7). Hence both half-periods T_h and T_H are intrinsically real and represent the oscillatory character of the motion in the associated regions.

As usual, one would call the transition of the particle from the left ($x \leq b$) of the barrier to the right ($x \geq c$) as tunneling. The classical considerations we have followed till now do not provide a means for particle to transit from h -region to H -region and vice versa. For tunneling to occur, as is phenomenologically the case, there should be a cause for particle be hopping from the h -region trajectory to the H -region trajectory, which we necessarily attribute to quantum effects to be discussed below.

2.2 Quantum Considerations

In this section we shall discuss the issue of quantization to understand the tunneling effect. Let us first consider a particle in h -region whose position and momentum at time t are represented by operators $\hat{x}(t)$ and $\hat{p}(t)$, respectively. These operators do satisfy the fundamental quantization postulate

$$[\hat{x}(t), \hat{p}(t)] = i\hbar. \quad (14)$$

In the position basis, for example, the representation $\hat{x} = x$ and $\hat{p} = -i\hbar \frac{\partial}{\partial x}$ satisfies the quantum bracket (14). The Hamiltonian operator $\hat{H} = \hat{p}^2/2m_0 + V(\hat{x})$, after making the position basis replacements for $\hat{x}(t)$ and $\hat{p}(t)$, leads us to the Schroedinger equation for $\psi(x, t)$

$$\left\{ -\frac{\hbar^2}{2m_0} \frac{\partial^2}{\partial x^2} + V(x) \right\} \psi(x, t) = i\hbar \frac{\partial}{\partial t} \psi(x, t). \quad (15)$$

The stationary state solution (as is natural for a time-independent $V(x)$) for a slowly varying potential is the well known WKB wavefunction

$$\psi(x, t) \sim \frac{1}{\sqrt{p(x)}} \exp \frac{i}{\hbar} (-Et + \int^x dx' p(x')) \quad (16)$$

where $p(x) = \sqrt{2m_0(E - V(x))}$. As expected wavefunction is of oscillatory character, and diverges at the turning point as a by-product of the WKB approximation.

Now let us discuss the issue of quantization for the corpuscular representation for which the main object of the discussion is the energy function H (7). First of all, H leads to the Hamiltonian function $-p^2/2m_0 + V(x)$ where $p = -m_0 \frac{dx}{dt}$. If we quantize this Hamiltonian with the usual quantization prescription (14), we necessarily obtain a wrong sign kinetic energy operator $\frac{\hbar^2}{2m_0} \frac{\partial^2}{\partial x^2}$ which, when substituted in the Schroedinger equation (15), leads to an oscillatory wavefunction like (16), except for $p(x) \rightarrow \sqrt{2m_0(V(x) - E)}$. Such a solution is obviously unphysical because transition probability is expected to have an exponential fall-off with the barrier width as has already been confirmed by the experiment in various circumstances. Therefore, to overcome this difficulty one should find a way out, namely, one should apply

an appropriate quantization procedure in treating the particle in corpuscular representation. In searching for the appropriate quantization rule, one notices that Hamiltonian function for the corpuscular representation would have the same form as that of the wave representation if the momentum is allowed to take imaginary values $p \rightarrow ip$. Following the clue provided by this observation, the position and the momentum of the particle in the corpuscular representation are represented, respectively, by the operators $\hat{x}(t)$ and $\hat{p}(t)$ subject to the quantization rule

$$[\hat{x}(t), \hat{p}(t)] = \hbar. \quad (17)$$

Then, in the position basis the representation $\hat{x} = x$ and $\hat{p} = -\hbar \frac{\partial}{\partial x}$ satisfies this quantization condition. Obviously, momentum operator is anti-hermitian and its consequences will be discussed below. Using this position basis representation we obtain the Schroedinger equation for $\psi(x, t)$

$$\left\{ -\frac{\hbar^2}{2m_0} \frac{\partial^2}{\partial x^2} + V(x) \right\} \psi(x, t) = i\hbar \frac{\partial}{\partial t} \psi(x, t). \quad (18)$$

whose stationary state solution for a slowly varying potential is the well-known under-barrier WKB wavefunction

$$\psi(x, t) \sim \frac{1}{\sqrt{\tilde{p}(x)}} \exp \frac{i}{\hbar} (-Et + i \int^x dx' \tilde{p}(x')) \quad (19)$$

where $\tilde{p}(x) = \sqrt{2m_0(V(x) - E)}$. It is obvious that this wavefunction has non-oscillatory character in space and has the same form as one would obtain from the usual quantization procedure. This completes the quantization issue of wave- and corpuscule-like representations of the matter which have direct relevance to the tunneling phenomenon. Now we shall turn to a comparative discussion of the present formalism and the usual one to point out the extra insight brought by the corpuscular representation.

3 Discussions and Conclusions

We now compare the results of this work with those of the usual formalism in a comparative manner. We first summarize the discussion of the tunneling phenomenon in usual terms and touch on some important points both in classical and quantum mechanical regimes.

1. In the usual formalism one starts, at the classical level, with the energy function $\frac{1}{2}m_0(\frac{dx}{dt})^2 + V(x)$ which is valid everywhere provided that the associated Lagrange function yields the correct Newtonian laws of motion. As the Hamiltonian does not have an explicit time dependence it is conserved, and the total energy E is a constant of motion. As discussed in detail in Sec. 2.1, in this case particle experiences an oscillatory motion with a period twice (12). In those portions of space where $V(x) \geq E$, momentum of the particle $\sqrt{2m_0(E - V(x))}$ turns to imaginary and for consistency one needs to make $m_0\frac{dx}{dt}$ imaginary. As a natural choice, one continues time to pure imaginary values: $t \rightarrow -it$. Therefore, at the classical level of the discussion tunneling event could be concluded to cost no real time at all.

2. In the above-mentioned classical setting, one applies the quantization rule (14) and derives the associated wave equation for $\psi(x, t)$ which is valid everywhere in space-time. In doing this one assumes that -referring the discussion at the beginning of Sec.2- the rest energy of a (masive) particle is equivalent to an amount of the Planckian energy which is always associated with a wave propagation- as we named in the text 'wave representation'. The Schrodinger equation, which is a differential equation that that wave motion is to satisfy in the non-relativistic interacting limit, describes the dynamics of the particle everywhere. In the tunneling region, where $V(x) \geq E$, it produces evanescent waves which exponentially decay (or grow) in space. A similar situation occurs in solving the Maxwell equations, for example in conducting media, and is a sign of the power loss in the medium. In such cases one does not have a wave propagation, instead a localized distribution showing the fact that (in the case of quantum mechanics) it will be less and less probable to find the particle in a certain portion of space if one goes away and away from the point of entrance into the barrier region.

Now we turn to the discussion of the tunneling time [9, 14]. As is well known, time is not representable by an operator in quantum theory, and there is no unique way of calculating how long it will take for the particle to appear at the opposite side of the barrier. We have discussed nature of the classical time for the barrier region in Item 1: independent of the parameters of the particular problem it is purely

imaginary; saying that tunneling occurs in zero real time, modulo the will-be quantum mechanical contribution.

Now we start discussing the implications of this work about the mechanism of the tunneling phenomenon. According to the present work we deal with two sets of relativistic transformation laws. The first group applies all the observable systems and have already been confirmed phenomenologically. In this case one naturally arrives at the well-known equations describing the Lorentz transformations, non-relativistic equations of motion, and wave-particle duality. We called this type of interpretation as 'wave representation' in the text.

The second group applies no observed system and includes strange laws, for example, energy of a system decreases with its speed and vanishes when it reaches the speed of light. In this case, the frequency in the Plackian energy is an internal property of the system itself, and one does not have a wave propagation coexisting with the particle at all. In fact, the system at rest is a clock tickling with a frequency proportional to its rest mass (1). Transformation to a moving frame just rescales the rest-frame parameters without leading one to a four-vector relation as in the wave representation. We named this representation as the 'corpuscular representation' in the text.

Needless to say, two transformation laws become identical in the rest frame. This work proposes that the non-relativistic interacting Hamiltonian for a (massive) particle should be written as in the first group of laws when it is in normal regions where Newton's equation of motion is valid, and as in the second group when it is not.

The quantum tunneling provides a unique chance where the above classification may find a place of application. This can be understood by the following observations: Particles outside the barrier region are under direct observation and have phenomenologically well-established properties, so that here one can apply the laws of motion residing in the first group. Next, at the classical turning points particle is momentarily at rest for which one has the situation described in (1). Finally, away from the turning point, particle may either return back to the normal region or enter the barrier region in which about the dynamics of the particle one does not have direct information. Hence, proposing that particle in the barrier region can be described by the second group of laws - as long as consistent - is not forbidden a priori.

1. Unlike the usual case which ascribes a single Hamiltonian function following from (6), in accordance with these arguments we assume equations (6) and (7) to describe the dynamics of the particle outside and inside the barrier, respectively. As discussed in detail in Sec. 2.1, under such a prescription one arrives at classical trajectories resting exclusively on real-time trajectories in both normal (8) and barrier (10) regions. Despite the purely real nature of the time parameter, one obtains the same classical equations of motion (see (9) and (11)) appearing in the usual discussion of the problem. Thus at the classical level of the discussion one concludes that time remains real for both normal (as it should) and barrier regions.
2. In quantizing this classical setting, one adopts the quantization prescriptions (14) and (17) for Hamiltonians in normal and barrier regions, respectively. Like the Hamiltonian functions themselves, the quantization prescriptions do also vary as one changes from normal to barrier regions. These quantizations rules do naturally lead us to the Schroedinger equation appropriate for the region under consideration (see (15) and (18)).

Unlike the 'evanescent wave' characterization of the usual discussion, the exponentially decaying character of the under-barrier wavefunction can now be attributed to the corpuscular nature of the particle consistent with the discussions in the previous sections. One can envisage the appearance of the 'corpuscular' nature of the particle as its effective localization over one de Broglie wavelength. That one arrives at the same wavefunction (19) as one would obtain by the known methods is a mathematical statement, physically (19) can be interpreted as the quantum generalization of the classical material point. This can be seen by observing that as $\hbar \rightarrow 0$, $\psi(x, t)$ in (19) vanishes indicating the fact that particle returns back to its classical representation of vanishing extension (a material point) and is pushed outside the barrier region; the only place (\hbar -region) it is allowed to exist classically. Thus the interpretation of the frequency ω_0 in (1) as some internal property of the particle itself leads one eventually to a quantum generalization of the classical material point. On the basis of these observations one would understand the mechanism of the tunneling phenomenon as occurring when the effective spatial extension of the particle becomes comparable

to the barrier width.

We now discuss nature of the momentum operator in two approaches. As tunneling occurs always with $V(x) \geq E$, it is meaningless to consider states with definite momentum, except for the idealized case of a semi-infinite flat potential for which the Schrodinger equation (18) has the stationary -state solution $\phi(x) = \text{constant} \times \exp -\frac{px}{\hbar}$, where $p = \sqrt{2m_0(V_0 - E)}$, and V_0 is the barrier hight. In the standard analysis of the tunneling effect momentum operator $-i\hbar\frac{\partial}{\partial x}$ is hermitian, and $\phi(x)$ is an its eigenfunction with the eigenvalue ip . On the other hand, in the present approach, momentum operator $-\hbar\frac{\partial}{\partial x}$ is not hermitian, and $\phi(x)$ is an its eigenfunction with the eigenvalue p . Thus, we conclude that in the present approach momentum operator reproduces the decaying and growing exponentials with real eigenvalues, whereas in the usual approach one should introduce purely imaginary eigenvalues to obtain the same eigenfunctions.

Finally we comment on the tunneling time. The amount of time a particle spends to traverse a given potential barrier cannot be evaluated in the formalism of this work. Then, it remains to apply to certain phenomenological methods developed already [14].

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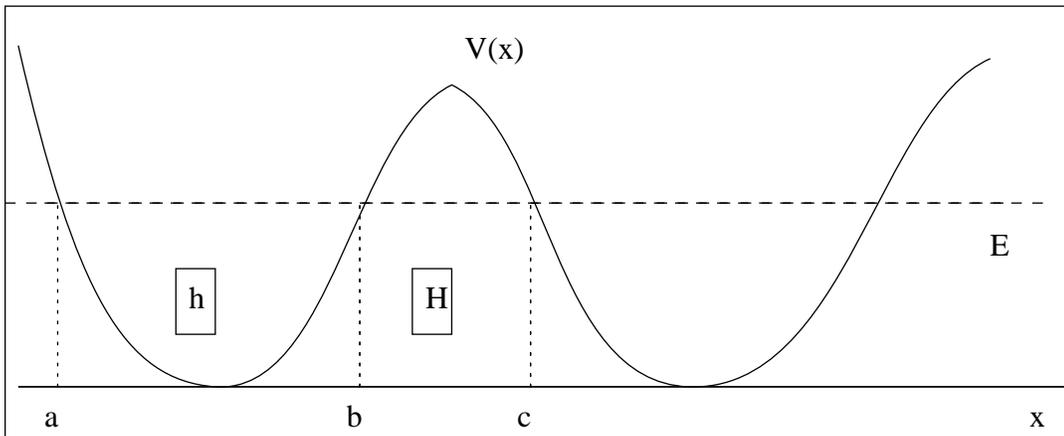


Figure 1: A typical potential realizing the quantum mechanical tunneling (see text).